Iwasawa Theory

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This course will provide an introduction to an active area of research in algebraic number theory. One of the central concerns of algebraic number theory is the understanding of the structure of the class group of a number field. Iwasawa theory is concerned with how these groups grow as one climbs a tower of number fields. It turns out that this growth is quite regular, and can largely be described in terms of certain invariants. In turn, this growth is linked to analytic functions known as $L$-functions, and especially their $p$-adic counterparts. Moreover, Iwasawa theory extends to study the growth of other interesting objects, such as Selmer groups of elliptic curves.

Prerequisites for the course include graduate algebra and algebraic number theory, hopefully but not necessarily including class field theory. Some familiarity with elliptic curves and modular forms is helpful, but not necessary. I will introduce the subject from my own unique perspective and intend to indicate recent directions in the field as the course progresses. Some of the fundamental topics that I expect to cover include:

- Dirichlet characters, Bernoulli numbers, $L$-functions, and $p$-adic $L$-functions
- Growth of arithmetic objects in towers and the structure theory of Iwasawa modules
- Classical theorems and conjectures on Galois groups of maximal abelian extensions unramified outside a set of primes
- The main conjecture of Iwasawa theory (which is a theorem)
- Key ideas of the two methods of proof of the main conjecture
- Basics of the Iwasawa theory of elliptic curves and Selmer groups