Algebra Qualifying Examination

January 2016

Do either one of \( nA \) or \( nB \) for \( 1 \leq n \leq 5 \). Justify all your answers.

1A. Let \( A \), \( B \) be two square matrices over a field \( F \). Suppose that \[
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix}
\]
is similar to \[
\begin{pmatrix}
B & 0 \\
0 & A
\end{pmatrix}
\]. Prove that \( A \) is similar to \( B \).

1B. Let \( F \) be an infinite field and let \( V \) be a \( F \)-vector space. Show that if \( V = \bigcup_{i=1}^{n} V_i \) for \( F \)-subspaces, \( V_1, \ldots, V_n \), then there is \( j \in \{1, \ldots, n\} \) with \( V = V_j \).

2A. Show that the automorphism group of the quaternion group of order 8 is a semidirect product of a group of order 4 and a group of order 6.

2B. Let \( G \) be a finite simple (abelian or non-abelian) group of order \( n \). Find the number of normal subgroups of \( G \times G \).

3A. Give a complete proof of the Hilbert Basis Theorem: If \( R \) is a commutative Noetherian ring with identity, then so is \( R[x_1, x_2, \ldots, x_n] \).

3B. Let \( R \) be a PID and \( I \) a nonzero ideal of \( R \). Show that there are only finitely many ideals of \( R \) containing \( I \). Show by example that this may not hold if \( R \) is a UFD but not a PID.

4A. Let \( F = \mathbb{C} \), let \( K = \mathbb{C}(t) \), the field of rational functions in an indeterminate \( t \), and let \( G \) be the Galois group \( G(K/F) \). Suppose \( \varphi \) and \( \theta \) in \( G \) are determined by \( \varphi(t) = \zeta t \) and \( \theta(t) = 1/t \), where \( \zeta \) is a primitive \( n \)th root of unity in \( \mathbb{C} \), \( n \geq 4 \), and set \( H = \langle \varphi, \theta \rangle \leq G \). Show that \( H \) is isomorphic with the dihedral group \( D \) of order \( 2n \) and show that the fixed field of \( H \) is \( \mathbb{C}(t^n + t^{-n}) \).

4B. Let \( \xi \in \mathbb{C} \) be such that \( \xi^{2015} = 3 \). Show that \( -3 \) is not a sum of squares in \( \mathbb{Q}(\xi) \).

5A. Let \( G \) be the group given by the presentation \( \langle x, y, z | x^2, y^3, (xyz)^4 \rangle \). Write the commutator factor group \( G/[G,G] \) as a direct product of cyclic groups and justify your answer.
5B. Let $A$ be a finite dimensional, associative (not necessarily commutative) $\mathbb{C}$-algebra with no zero divisors, but with identity. Show that $\dim_{\mathbb{C}} A = 1$. 