Analysis Qualifying Exam

PLEASE SHOW ALL YOUR WORK

Problem 1. For a measurable function $f : [0,1] \to \mathbb{R}$, let

$$g(x,y) = f(x) - f(\sqrt{y}).$$

Show that if $g(x,y) \in L^1([0,1] \times [0,1])$, then $f \in L^1([0,1])$.

Problem 2. Let $f : l^2(\mathbb{N}) \to \mathbb{R}^{\mathbb{N}}$ be defined by

$$f(\{x_n\}) = \{y_n\} \text{ where } y_n = x_n x_{n+1}.$$

(a) Show that f maps $l^2(\mathbb{N})$ to $l^1(\mathbb{N})$.

(b) Show that f is continuous. (Note that f is not linear.)

Problem 3. Let f_n be a sequence in $L^1(\mathbb{R})$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x) g(x) dx = g(0) \text{ for each } g \in C_0(\mathbb{R}).$$

(Here, $C_0(\mathbb{R})$ denotes the set of all continuous functions on $\mathbb{R}$ vanishing at infinity.) Show that f_n is not Cauchy in $L^1(\mathbb{R})$.

Problem 4. Let $f : \mathbb{R} \to [0,\infty]$ be a Lebesgue measurable extended real valued function. Define a measure μ by

$$\mu(E) = \int_E f dx.$$

Show that μ is σ-finite if and only if $|f(x)| < \infty$ Lebesgue a.e.

Problem 5. Consider the function $f : (0,\infty) \times (0,\infty) \to \mathbb{R}$ defined by

$$f(x,y) = \sum_{n=1}^{\infty} \frac{1}{nx^2 + n^3 x^{-2} y}.$$

Find the limit $g(y) := \lim_{x\to\infty} f(x,y)$ for $y > 0$, with a proof.

Problem 6. For A and B, subsets of $\mathbb{R}^2$, define $A + B = \{x + y : x \in A \text{ and } y \in B\}$.

(a) Let A and B be compact subsets of $\mathbb{R}^2$. Show that $A + B$ is compact.

(b) Let A and B be closed subsets of $\mathbb{R}^2$. Show that $A + B$ is not necessarily closed.