Problem 1. Let \( f_n(x) \geq 0 \) be continuous functions on \([0, 1]\). Suppose that \( \lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 0 \) and that for all \( x \), \( \lim_{n \to \infty} f_n(x) = 0 \). Prove or disprove that \( f_n \) must converge uniformly to 0 on \([0, 1]\).

Problem 2. Let \((X, \mu)\) be a \( \sigma \)-finite measure space. Prove that \( \mu(X) < \infty \) if and only if \( L^2(X, \mu) \subset L^1(X, \mu) \).

Problem 3. Let \( m \) be Lebesgue measure on \([0, 1]\). Given \( f \in L^2([0, 1], m) \) define
\[
Kf(x) = \frac{1}{x^{4/3}} \int_0^x f(t) \, dt.
\]
(a) Show that there is a constant \( C \) such that \( \|Kf\|_1 \leq C \|f\|_2 \) for all \( f \in L^2([0, 1], m) \), i.e., \( K \) is a bounded operator from \( L^2([0, 1], m) \) to \( L^1([0, 1], m) \).
(b) Find the operator norm of \( K \).

Problem 4. Let \((X, \mu)\) be a finite measure space. Let \( f \in L^1(X, \mu) \). For \( t \in \mathbb{R} \) define
\[
g(t) = \int_X \cos(tf(x)) \, d\mu(x)
\]
Prove that \( g(t) \) is differentiable for all \( t \) and that the derivative is a continuous function on \( \mathbb{R} \).

Problem 5. Let \( f_n \) be absolutely continuous functions on \([a, b]\), \( f_n(a) = 0 \) for all \( n \). Suppose \( f'_n \) is a Cauchy sequence in \( L^1([a, b], m) \) where \( m \) is the Lebesgue measure. Show that there exists an absolutely continuous function \( f \) on \([a, b]\) such that \( f_n \to f \) uniformly on \([a, b]\).

Problem 6. Let \( f, f_k : \mathbb{R} \to \mathbb{R} \) be Lebesgue measurable functions such that \( f_k \to f \) a.e. and there exists a Lebesgue integrable function \( g \) (\( g \in L^1(\mathbb{R}) \)) such that
\[
|f_k(x)| \leq g(x) \text{ a.e. for all } k.
\]
The goal in this problem is to prove that \( f_k \to f \) almost uniformly, i.e., for any \( \delta > 0 \) there exists \( E \subset \mathbb{R} \) such that \( m(E) < \delta \) and \( f_k \to f \) uniformly on \( E^c \). The measure \( m \) is Lebesgue measure. Let \( X_0 = \{x : g(x) = 0\} \) and \( X_n = \{x : |g(x)| \geq 1/n\} \) for \( n \in \mathbb{N} \) so that \( \mathbb{R} = \bigcup_{n=0}^{\infty} X_n \).

(a) Show that \( X_n \) has finite measure for all \( n \geq 1 \).
(b) Show that for any \( \delta > 0 \), there is a set \( E \) such that \( m(E) < \delta \) and for all \( n \geq 1 \) the sequence \( f_k \) converges to \( f \) uniformly on \( E^c \cap X_n \).
(c) Show that \( f_k \) converges to \( f \) uniformly on \( E^c \).