ANALYSIS QUALIFYING EXAM

PLEASE SHOW ALL YOUR WORK

GOOD LUCK!

PROBLEM 1

Let

\[ f(x) = \int_0^2 \frac{\sin t}{t^{3/2}} \, dt. \]

Find

\[ \lim_{k \to \infty} \int_0^\infty k^{3/2} f(x) e^{-kx} \, dx. \]

Justify all steps.

PROBLEM 2

Let \( f(x) \) be an absolutely continuous function on \([0, 1]\). Suppose that \( f(0) = 0 \).

Prove that

\[ \int_0^1 \frac{|f(x)|^3}{x^3} \, dx \leq \int_0^1 |f'(x)|^3 |\ln x| \, dx. \]

PROBLEM 3

Let \( x_n \) be a converging sequence of complex numbers. Suppose that

\[ \lim_{n \to \infty} x_n = 10. \]

Let

\[ y_n = \frac{1}{n^2} \sum_{k=1}^n kx_k. \]

Prove that the sequence \( y_n \) converges and find its limit.

PROBLEM 4

Let \( f \in L^1(\mathbb{R}) \), and let \( g, h \in L^2(\mathbb{R}, m) \). Prove that

\[ \int_{\mathbb{R}} |f(x)g(x-a)h(a)| \, dx < \infty \]

for almost every \( a \in \mathbb{R} \). Here \( m \) is the Lebesgue measure.
Problem 5

Prove that the operator $A$ defined by the formula

$$Af(x) = \int_0^1 \frac{f(y)}{\sqrt{|x-y|}} dy$$

is a bounded operator in $L^1([0,1], m)$ and find its norm; $m$ is the Lebesgue measure.

Problem 6

Let

$$f(x, y) = \max\{x^2 + y^2, 1\}.$$

Define a Borel measure $\mu$ on $\mathbb{R}$ by the formula

$$\mu(E) = (m \times m)(f^{-1}(E))$$

where $m$ is the Lebesgue measure. Find the Radon-Nikodym derivative $d\mu_{ac}/dm$ where $\mu_{ac}$ is the absolute continuous part of $\mu$. 