Please show all of your work. GOOD LUCK!

(1) A real-valued function $f$ on $[a, b]$ is said to be locally bounded if for every $x \in [a, b]$ there is a $\delta > 0$ such that $f(x)$ is bounded on $(x - \delta, x + \delta) \cap [a, b]$. Prove that if $f$ is locally bounded on $[a, b]$ then $f$ is bounded on $[a, b]$.

(2) i) Prove the following estimate:
$$\int_1^{\infty} \frac{\sqrt[3]{1 + x}}{x^2} \, dx \leq \sqrt[3]{6}.$$ ii) Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$. Let $f$ and $g$ be measurable functions on $X$ that are positive $\mu$-a.e. Suppose that $1 \leq f(x)g(x)$ for $\mu$-a.e. $x \in X$. Prove that
$$1 \leq \int_X f \, d\mu \cdot \int_X g \, d\mu.$$ 

(3) Let $\{f_n\}_{n \geq 1}$ be a sequence of measurable, real-valued functions on $[0, 1]$. i) Suppose there exists positive numbers $C$ and $\epsilon$ for which
$$\int_0^1 f_n(x)^2 \, dx \leq Cn^{2-\epsilon} \quad \text{for all } n \geq 1.$$ Show that $g_n = \frac{f_n}{n}$ converges to zero in measure. ii) Suppose $f_n = \chi_{E_n}$ with $E_n \subset [0, 1]$ for all $n \geq 1$. Show that if $f_n$ converges to $f$ in $L^1$, then $f$ is also the characteristic function of a measurable set.
(4) Let \( f \in L^1([0,1]) \) with \( f \notin L^2([0,1]) \). Consider the following:

\[
\lim_{n \to \infty} \int_0^1 n \ln \left(1 + \frac{|f(x)|^2}{n^2}\right) \, dx
\]

Determine whether or not this limit exists. If it exists, calculate it. If it does not, explain.

(5) Let \((X, \mathcal{M}, \mu)\) be a \(\sigma\)-finite measure space. Let \( f \) be a non-negative, measurable function on \( X \). Define a function on \([0, \infty)\) by

\[
F(t) = \mu(\{x : f(x) > t\})
\]

Prove that for all \( \alpha \geq 0 \), \( t^\alpha F(t) \) is integrable with respect to Lebesgue measure on \([0, \infty)\) if and only if \( f \in L^{1+\alpha}(X, \mu) \).

Hint: express \( F(t) \) as an integral.

(6) For non-negative integers \( k \), define

\[
S_k = \{f \in L^2([0,1]) : \int_0^1 f(x) \exp(2\pi i nx) \, dx = 0, \text{ if } |n| \leq k\}
\]

Let \( P_k \) be the orthogonal projection onto the closed subspace \( S_k \). Prove or disprove each of the following statements:

i) \( \|P_k\| \to 0 \) as \( k \to \infty \).

ii) \( \forall f \in L^2([0,1]) \), we have \( \|P_k f\| \to 0 \) as \( k \to \infty \).