Geometry-Topology Qualifying Exam

Fall 2015

Problem 1
Evaluate the integral

$$\int_0^{2\pi} \frac{\sin^2 3\phi}{5 - 4\cos 2\phi} d\phi.$$

Hint: You might want to use the substitution $z = e^{i\phi}$.

Problem 2
(a) Show that

$$M := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x^2 + xy + y^2 + xz + z^3 = 1 \right\}$$

is an embedded submanifold of $\mathbb{R}^3$.

(b) Compute the tangent space to M at $p = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ as a subspace of $\mathbb{R}^3$.

(c) For the function

$$f : M \to \mathbb{R} \tag{1}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto z \tag{2}$$

find the direction (tangent to M) of greatest increase at the same point p as in part (b).

Problem 3

Let X and Y be two vector fields and f and g two smooth functions. Prove that

$$[fX, gY] = fg[X, Y] - gY(f)X + fX(g)Y,$$

where $[\cdot, \cdot]$ denotes the Lie bracket of vector fields.

Problem 4

The space Y is obtained by removing a small disk from a two-torus and identifying the boundary of the resulting space with a torus meridian, see Figure 1. Find the fundamental group $\pi_1(Y)$ (at a basepoint of your choice).

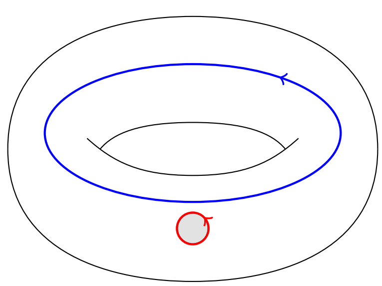

Figure 1: Punctured torus with meridian (top, in blue) and deleted disk (shaded, with boundary in red).

Problem 5

Let X be a connected sum of the Klein bottle and a real projective space, i.e.

- let K' denote the Klein bottle with a small open disk removed, and let $f : \partial K' \to S^1$ be a homeomorphism of the boundary of K' with the circle;
- let P' denote the real projective plane with a small open disk removed, and let $g : \partial P' \to S^1$ be a homeomorphism;
- then X is obtained by identifying the boundary of K' with that of P' :

$$X = K' \coprod P' / \sim,$$

where $x \sim y$ if $x \in \partial K'$, $y \in \partial P'$ and $f(x) = g(y)$, and X has the quotient topology.

a) Compute the homology groups of X with integer coefficients.

b) Infer the cohomology groups of X with integer coefficients (or compute them directly).