GEOMETRY-TOPOLOGY QUALIFYING EXAM, AUGUST 2016

1. Let $H^2 = \{\tau \in \mathbb{C} : Im(\tau) > 0\}$, the upper half of the complex plane. Consider the map

$$P : H^2 \to \mathbb{C} : \tau \mapsto q = exp(2\pi i \tau)$$

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(a) Find the image of the map P.

The map P defines a covering space for the image (you do not need to show this).

(b) Find the group of automorphisms of the covering space.

(c) Find a fundamental domain for this covering space.

2. (a) Determine the critical points and critical values for the function

$$f : \mathbb{R}^3 \to \mathbb{R} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto x^3 + y^2 - yz - y$$

(b) Let $M = f^{-1}(0)$. Find an explicit smooth parameterization for an open neighborhood of $\vec{0} \in M$.

(c) Find a basis for the tangent space of M at the point $\vec{0}$, viewed as a subspace of $\mathbb{R}^3$.

3. Consider the one-point compactification of $\mathbb{R}$, $\mathbb{R} \cup \{\infty\}$ (which is topologically equivalent to S^1). The compactification is a manifold, if we use x as a coordinate for $\mathbb{R}$ and $y = 1/x$ as a coordinate for the complement of 0 in the compactification.

(a) Show that $v = x\frac{\partial}{\partial x}$ extends to a smooth vector field on the compactification.

(b) Draw a qualitative picture of the flow lines for v, viewing the compactification as a circle.

(c) Recall that for stereographic projection $S^1 \to \mathbb{R} \cup \{\infty\}$

$$x = \frac{\cos(\theta)}{1 - \sin(\theta)}$$

Express the vector field v in terms of the θ coordinate. Does this agree with your work in (b)?

4. Let ω denote the differential form defined on $X = \mathbb{R}^2 \setminus \{0\}$ by

$$\omega = \frac{-ydx + xdy}{x^2 + y^2}$$

(a) Show that ω is closed in X.

(b) Show that ω is not exact in X.

(c) Let f be defined by $f(u,v) = (u^2 - v^2, 2uv)$. Calculate $f^*\omega$.

5. Compute the integral homology, and the DeRham cohomology, of the surface $(S^1 \times S^1) \setminus \{q\}$ (where q is a point on the torus). Briefly justify each calculation, and in particular indicate explicit generators for each class (for convenience you can view the torus as the quotient $\mathbb{R}^2/\mathbb{Z}^2$, and pick the point in any way you wish).

6. For each of the following statements, either briefly explain why the statement is true, or give a counterexample.

(a) Every exact k-form on a compact orientable k-dimensional manifold vanishes at some point.

(b) If the degree of a smooth map $f : S^2 \to S^2$ is nonzero, then the map f is onto.

(c) The Euler characteristic of a compact orientable 3-manifold is zero.