Problem 1

Find a conformal mapping from the strip \( \{ z : 0 < \text{Im} z < 1 \} \) to the first quadrant \( \{ \zeta : \text{Re} \zeta > 0, \text{Im} \zeta > 0 \} \) of the complex plane.

Problem 2

Let \( X \) and \( Y \) be manifolds, and let \( C^\infty_0(X) \) and \( C^\infty_0(Y) \) be spaces of smooth, compactly supported functions on \( X \) and \( Y \), respectively. Let \( p : X \to Y \) be a smooth covering. For \( f \in C^\infty_0(X) \), the function \( p_* f \) on \( Y \) is defined by the formula

\[
p_* f(y) = \sum_{x \in p^{-1}(y)} f(x).
\]

Prove that \( p_* \) is a surjective map from \( C^\infty_0(X) \) to \( C^\infty_0(Y) \).

Problem 3

Let

\[
X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}
\]

be a vector field in \( \mathbb{R}^2 \). What condition a differential form \( \omega = a(x, y)dx \wedge dy \) must satisfy for \( L_X \omega = 0 \)? Here \( L_X \) is the Lie derivative. Suppose that \( a(x, y) \) is not identically equal to 0. Can it be smooth? Give an example of a smooth, non-zero two-form \( \omega \) in \( \mathbb{R}^2 \setminus \{0\} \) such that \( L_X \omega = 0 \).

Problem 4

Let \( GL_+(2, \mathbb{R}) \) be the space of all \( 2 \times 2 \) matrices with real entries the determinant of which is positive. Compute \( \pi_1(GL_+(2, \mathbb{R})) \).

Problem 5

Let \( a_1, \ldots, a_k \) be \( k \) distinct points on the two-dimensional sphere \( S^2 \), and let \( X = S^2 \setminus \{a_1, \ldots, a_k\} \). Compute singular homologies of \( X \) with integer coefficients.

Problem 6

Compute \( \pi_1(\mathbb{R}P^2 \times S^1) \). Here \( \mathbb{R}P^2 \) is the space of all lines passing through the origin in \( \mathbb{R}^3 \), and \( S^1 \) is the unit circle in \( \mathbb{R}^2 \).