1. Sequences and Series

**Def 1.1** A set $M$ and a function $d : M \times M \to \mathbb{R}^+$ are called a metric space if

1. $d(x, y) = d(y, x)$
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)

**Def 1.2** A sequence $\{x_n\}$ converges to $x$, if for every $\epsilon > 0$ there exists $N$, such that for all $n \geq N$,

$$d(x, x_n) \leq \epsilon.$$ 

**Def 1.3** A sequence $\{x_n\}$ is called Cauchy (or fundamental) if for every $\epsilon > 0$ there exists $N$, such that

$$d(x_m, x_n) \leq \epsilon, \quad \text{for all} \quad m, n \geq N.$$

**Def 1.4** A metric space is called complete if every Cauchy sequence converges.

**Def 1.5** A metric space is called compact if any sequence has a converging subsequence.

**Def 1.6** Series

$$\sum_{n=1}^{\infty} x_n$$

(1)

converges if its partial sums, $S_N = \sum_{n=1}^{N} x_n$ converge as $N \to \infty$.

**Def 1.7** Series (1) converges absolutely if

$$\lim_{N \to \infty} \sum_{n=1}^{N} |x_n| < \infty.$$ 

**Def 1.8** Given a power series, $\sum_{n=0}^{\infty} c_n z^n$, define

$$\alpha = \limsup_{n \to \infty} \sqrt[n]{|c_n|}; \quad R = \frac{1}{\alpha}.$$ 

$R$ is called the radius of convergence of the series, the latter converges if $|z| < R$ and diverges if $|z| > R$. 

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1.1. Convergence Tests

**Root test.** Let \( \alpha = \limsup_{n \to \infty} \sqrt[n]{|x_n|} \). Series (1) converges (absolutely) or diverges if \( \alpha < 1 \) or \( \alpha > 1 \) respectively. (More analysis is required if \( \alpha = 1 \))

**Ratio test.** Series (1) converges (absolutely) if \( \limsup_{n \to \infty} \left| \frac{x_{n+1}}{x_n} \right| < 1 \) and diverges if there exists some number \( N \) such that \( \left| \frac{x_{n+1}}{x_n} \right| \geq 1 \) for all \( n > N \).

**Comparison test.** If \( \lim_{n \to \infty} \frac{x_n}{y_n} = C \in (0, \infty) \), series \( \sum_{n=1}^{\infty} x_n \) converges absolutely iff series \( \sum_{n=1}^{\infty} y_n \) does.

1.2. Is There a “Boundary” Between Converging and Diverging Series?

The series
\[
\sum_{n=1}^{\infty} \frac{1}{n^\alpha}
\]
converges for \( \alpha > 1 \) and diverges for \( \alpha \leq 1 \). Thus the exponent \( \alpha = 1 \) corresponds to the “boundary” for power-law decay rates between converging and diverging series. However, for more general functions, how “close” can we get to \( 1/n \) while still maintaining convergence? For example,
\[
\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^\alpha}
\]
converges for all \( \alpha > 1 \) and diverges for \( \alpha \leq 1 \). So we lifted our boundary a bit, from \( 1/n \) to \( 1/(n \ln n) \). We can go even further and observe that
\[
\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln \ln n)^\alpha}
\]
converges for all \( \alpha > 1 \) and diverges for \( \alpha \leq 1 \). Etc, etc: we can keep adding more iterated logarithms (or other functions) in a similar manner. Is there some limit to this process? In other words, e.g., is there some special monotone-decreasing sequence \( \{ b_n \} \) such that whenever \( c_n/b_n \to 0 \) (as \( n \to \infty \)) the series \( \sum c_n \) converges and whenever \( b_n/d_n \to 0 \), the series \( \sum d_n \) diverges?

1.3. Fun Stuff

Consider the geometric series,
\[
\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad |z| < 1.
\]

Pretending that this formula is valid for arbitrary \( z \neq 1 \), we can “derive” that \( 1 - 1 + 1 - 1 + \cdots = 1/2 \), or \( 1 + 2 + 3 + 4 + 8 + \cdots = -1 \). In this case the divergent sum acquires meaning via analytic continuation of some appropriately chosen function outside of the radius of convergence of its power series. In a similar fashion one can get such formulas as, e.g.,
\[
1 - 2 + 3 - 4 + \cdots := \left. \frac{1}{(1+z)^2} \right|_{z=1} = \frac{1}{4}, \quad 1 + 2 + 3 + 4 + \cdots := \zeta(-1) = -\frac{1}{12}.
\]
2. Continuity and Differentiation

Unless specified otherwise, we consider functions between metric spaces $\mathcal{X}$ and $\mathcal{Y}$.

**Def 2.1** A function $f$ is called **continuous at** $x_0$ if for every $\varepsilon > 0$ there exists $\delta > 0$, such that for all $x \in \mathcal{X}$ with $d_{\mathcal{X}}(x, x_0) < \delta$, $d_{\mathcal{Y}}(f(x), f(x_0)) < \varepsilon$. A function which is continuous at every point of $\mathcal{X}$ is called **continuous in** $\mathcal{X}$.

**Def 2.2** A function $f$ is called **uniformly continuous** if for every $\varepsilon > 0$ there exists $\delta > 0$, such that for all $x_1, x_2 \in \mathcal{X}$ with $d_{\mathcal{X}}(x_1, x_2) < \delta$, $d_{\mathcal{Y}}(f(x_1), f(x_2)) < \varepsilon$.

Assume that our metric spaces are also **normed** linear vector spaces with norm defined as $\|f\| = d(f, 0)$.

**Def 2.3** Suppose $\mathcal{O}$ is an open set in $\mathcal{X}$; $f$ maps $\mathcal{O}$ into $\mathcal{Y}$; $x_0 \in \mathcal{O}$. If there exists a **bounded** linear operator $Df(x_0)$, such that

$$\lim_{\|x\|_{\mathcal{X}} \to 0} \frac{\|f(x_0 + x) - f(x_0) - Df(x_0)x\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}} = 0,$$

(2)

then $f$ is called **differentiable at** $x_0$, and $Df(x_0)$ is called the (Fréchet) derivative or differential of $f$ at $x_0$. If $f$ is differentiable at every point in $\mathcal{O}$, we call $f$ differentiable in $\mathcal{O}$. The determinant of the operator $Df(x_0)$ (if well-defined) is called the **Jacobian** of $f$ at $x_0$.

2.1. Some Important Results

**Mean value theorem.** Suppose $f$ is continuous on $[a, b]$ and differentiable in $(a, b)$. There exists $x \in (a, b)$, such that

$$f'(x) = \frac{f(a) - f(b)}{a - b}.$$**

**Taylor’s theorem (1d).** Suppose $f \in C^{n-1}[a, b]$ and $f^{(n)}(x)$ exists for all $x \in (a, b)$. For all $x$ and $y$ such that $a < x < y < b$, there exists $\zeta \in [x, y]$ such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(y)}{k!} (x - y)^k + \frac{f^{(n)}(\zeta)}{n!}(x - y)^n.$$**

**Taylor’s theorem (multi-d).** Let $\mathcal{B}$ be a closed ball centered at the origin in $\mathbb{R}^n$; $f \in C^n(\mathcal{B})$; $x \in \mathcal{B}$. Then

$$f(x) = \sum_{|\alpha| < n} \frac{x^\alpha}{\alpha!} \partial^\alpha f(0) + \sum_{|\alpha| = n} \frac{x^\alpha}{\alpha!} \partial^\alpha f(\xi), \text{ for some } \xi \in [0, 1].$$**

**Inverse function theorem.** Assume that $f$ is a continuously differentiable function from $\mathbb{R}^n$ and $Df(x)$ is invertible. Then $f$ is invertible in some neighborhood of $x$ and its inverse is continuously differentiable in some neighborhood of $f(x)$.

**Implicit function theorem.** Assume that $F$ is a continuously differentiable function from (an open subset) $\mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}^m$; $(x, y) \in \mathcal{O}$; $F(x, y) = 0$; and $Df$ is one-to-one. Then there exists a neighborhood $\mathcal{N} \subset \mathbb{R}^n$ containing $x$ and a function $f : \mathcal{N} \to \mathbb{R}^m$, such that $f(x) = y$ and $F(x, f(x)) = 0$ for all $x \in \mathcal{N}$.
3. Integration, Theorems Relating Integrals and Derivatives

**Def 3.1** A finite ordered subset of \([a,b], \pi = (\pi_1, \ldots, \pi_n)\), such that
\[
a = \pi_1 < \pi_2 < \ldots < \pi_{n-1} < \pi_n = b
\]
is called a partition of \([a,b]\). We say \(\pi_2\) is a refinement of \(\pi_1\) if \(\pi_1 \subset \pi_2\). A sequence of partitions \(\{\pi^n\}\) is called fine if each partition in the sequence is a refinement of the previous one and
\[
\lim_{n \to \infty} \max_{k=2,\ldots,|\pi^n|} (\pi^n_k - \pi^n_{k-1}) = 0.
\]

**Def 3.2** Suppose the functions \(f\) and \(g\) are such that following limit exists and is the same for all fine sequences of partitions of \([a,b]\) and all \(x(\pi) = (x_2, \ldots, x_{|\pi|})\) such that \(x_k \in [\pi_{k-1}, \pi_k], k = 2, \ldots, |\pi|:\)
\[
\lim_{n \to \infty} \sum_{k=2}^{|\pi^n|} f(x_k(\pi^n)) |g(\pi^n_k) - g(\pi^n_{k-1})|.
\]
It is then called the Riemann-Stieltjes integral of \(f\) with respect to \(g\) over \([a,b]\) = \(\Omega\) and is denoted by
\[
\int_a^b f(x) \, dg(x) \quad \text{or} \quad \int_{\Omega} f \, dg. \tag{3}
\]

- If \(g\) is differentiable, then Riemann-Stieltjes integral can be related to the usual Riemann integral,
\[
\int_{\Omega} f \, dg = \int_{\Omega} f(x) g'(x) \, dx.
\]

**Def 3.3** A function \(f : [a,b] \to \mathbb{R}\) is called bounded variation if
\[
V_b^a(f) := \sup_{\pi \in \mathcal{P}[a,b]} \sum_{n=2}^{|\pi|} |f(\pi_n) - f(\pi_{n-1})| < \infty.
\]
Here \(\mathcal{P}[a,b]\) denotes the set of all partitions of \([a,b]\). The space of all functions of bounded variation on \([a,b]\) is denoted by \(BV[a,b]\).

3.1. Some Important Results

**Existence of Riemann-Stieltjes integral** Suppose \(f \in C[a,b]\) and \(g \in BV[a,b]\), then the Riemann-Stieltjes integral (3) exists.

- For a given \(g \in BV[a,b]\), the class of functions integrable with respect to \(g\) is larger than \(C[a,b]\) and essentially includes all Riemann-integrable functions which do not share points of discontinuity with \(g\).

**Fundamental theorem of calculus.** Let \(f \in C[a,b], g \in BV[a,b]\), then
\[
\int_a^b dg = g(b) - g(a); \quad \text{if in addition} \ g \in C[a,b], \ \text{then} \ \frac{d}{dg} \int_a^x f(y) \, dg(y) = f(x) \quad \text{for all} \ x \in [a,b].
\]
Here \(\frac{dF(x)}{dg(x)} := \lim_{\epsilon \to 0} \frac{F(x+\epsilon) - F(x)}{g(x+\epsilon) - g(x)}\) — (essentially) the Radon-Nikodym derivative of \(F\) with respect to \(g\).
Change of variables. Suppose \( g, h \in \text{BV}(\Omega) \); \( f, \frac{dg}{dh} \in \mathbb{C}[a, b] \), then
\[
\int_{\Omega} f \, dg = \int_{\Omega} f \frac{dg}{dh} \, dh.
\]

Integration by parts. Suppose \( f, g \in \text{BV}[a, b] \), \( f \in \mathbb{C}[a, b] \), then
\[
\int_{a}^{b} f \, dg = f(b)g(b) - f(a)g(a) - \int_{a}^{b} g \, df.
\]

Integral mean value theorem I. Let \( f \) be continuous and \( g \) monotone on \([a, b]\), then there exists \( x \in [a, b] \), such that
\[
\int_{a}^{b} f \, dg = f(x)[g(b) - g(a)].
\]

Integral mean value theorem II. Let \( f \) be monotone and \( g \) be continuous on \([a, b]\), then there exists \( x \in [a, b] \), such that
\[
\int_{a}^{b} f \, dg = f(a)[g(x) - g(a)] + f(b)[g(b) - g(x)].
\]

### 4. Sequences of Functions

**Def 4.1** A sequence of functions \( \{f_n\} \) converges to \( f \) **point-wise** in \( \mathcal{X} \) if for every \( x \in \mathcal{X} \),
\[
\lim_{n \to \infty} f_n(x) = f(x).
\]

**Def 4.2** A sequence of functions \( \{f_n\} \) converges to \( f \) **uniformly** in \( \mathcal{X} \) if for every \( \epsilon > 0 \) there exists \( N \) such that for all \( n > N \) and all \( x \in \mathcal{X} \),
\[
d(f_n(x), f(x)) < \epsilon.
\]

**Def 4.3** A family of functions, \( \mathcal{F} \), is called **equicontinuous** if for all \( \epsilon > 0 \) there exists \( \delta > 0 \), such that whenever \( d_{\mathcal{X}}(x_1, x_2) < \delta \),
\[
d_{\mathcal{Y}}(f(x_1), f(x_2)) < \epsilon \quad \text{for all} \quad f \in \mathcal{F}.
\]

### 4.1. Some Important Results

**Weierstrass M-test.** If \( \sup_{x \in \mathcal{X}} |f_n(x)| < M_n \) and the series \( \sum M_n \) converges, then \( \sum f_n(x) \) converges uniformly in \( \mathcal{X} \).

**Uniform convergence theorem.** A uniform limit of continuous functions is continuous.

**Monotone convergence theorem.** A point-wise monotone sequence of continuous functions converging to a continuous function on a compact set does so uniformly.

**Exchanging the order of limits and integration.** Suppose \( f_n \) converge uniformly to \( f \) in \( \Omega \) and each \( f_n \) is integrable with respect to \( g \) over \( \Omega \), then
\[
\lim_{n \to \infty} \int_{\Omega} f_n \, dg = \int_{\Omega} f \, dg.
\]
Exchanging the order of limits and differentiation. Suppose $f'_n$ converge uniformly on $[a, b]$ and $f_n$ converge at some $x_0 \in [a, b]$, then $f_n$ converge uniformly on $[a, b]$ to some differentiable function $f$ and
\[
\lim_{n \to \infty} f'_n(x) = f'(x).
\]

Stone-Weierstrass theorem. Continuous functions on $\mathbb{R}^n$ may be uniformly approximated by polynomials on compact subsets of $\mathbb{R}^n$.

Arzelà-Ascoli Theorem. Every infinite equicontinuous family of maps between compact metric spaces contains a uniformly converging sequence.