Analysis Problems
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Basic concepts, sequences, convergence

1. Every bounded increasing sequence in \( \mathbb{R} \) is Cauchy (do not use completeness of \( \mathbb{R} \)). Every bounded sequence in \( \mathbb{R}^n \) has a converging subsequence.

2. A metric space is compact if and only if it is complete and totally bounded, i.e., for every \( r > 0 \), it may be covered by a finite number of open balls of radius \( r \).

3. Every open set in \( \mathbb{R} \) is a union of at most a countable number of disjoint open intervals.

4. Compute \( \lim_{n \to \infty} \sqrt{n} \) without explicitly using that \( (\ln n)/n \to 0 \) as \( n \to \infty \).

Series

5. A conditionally convergent series (i.e., a non-absolutely convergent series whose partial sums still converge) may be summed to any desired number by an appropriate rearrangement of its terms.

6. A sequence is called Cesàro summable if the arithmetic means of its partial sums converge. What is the value of the sum \( 1 - 1 + 1 - 1 + \cdots \) in Cesàro sense?

7. Find examples of converging and diverging series for which \( \lim_{n \to \infty} |x_{n+1}/x_n| = 1 \) and \( \lim_{n \to \infty} \sqrt[n]{|x_n|} = 1 \).

8. If the coefficients of a power series are integers, infinitely many of which are nonzero (i.e., the series is not a polynomial) then the radius of convergence of this series is at most 1.

9. Prove that the radii of convergence of power series \( \sum_{n=0}^{\infty} a_n x^n \) and \( \sum_{n=1}^{\infty} n a_n x^{n-1} \) are the same.

10. Suppose all \( x_n \geq 0 \) and the series \( \sum_{n=0}^{\infty} x_n \) converges. Set \( y_n = \sum_{m=n}^{\infty} x_m \). Prove that \( \sum_{n=0}^{\infty} x_n/\sqrt{y_n} \) converges.

11. Prove that \( \sum_{n=1}^{\infty} x_n \) converges iff \( \prod_{n=1}^{\infty} (1 + x_n) \) converges.

12. Prove that \( e \) is irrational. (Hint: estimate approximation errors for partial sums of a series representation of \( e \) or some appropriate quantity related to it.)
CONTINUITY

13. Prove that a map of a metric space into a metric space is continuous iff pre-image of every open set is open. Show that if we replaced “pre-image” by “image,” the statement would be wrong.

14. An image of a compact set under a continuous map is compact.

15. Intermediate value theorem. An image of a connected set under a continuous map is connected. (A set is called connected if it cannot be represented as a union of disjoint nonempty open sets.)

16. Level sets of continuous functions are closed.

17. A function continuous on a compact set \( A \) is also uniformly continuous on \( A \).

18. A function \( f : \mathcal{X} \to \mathbb{R} \) is called convex if inequality,

\[
f(a_1x_1 + (1-a)x_2) \leq af(x_1) + (1-a)f(x_2)
\]

holds for all \( x_1, x_2 \in \mathcal{X} \) and \( a \in [0,1] \). Prove that convex functions are continuous.

19. A function \( f : \mathcal{X} \to \mathcal{Y} \) is called Hölder continuous with exponent \( \alpha \in [0,1] \) if there exists some constant \( C \) such that for all \( x_1, x_2 \in \mathcal{X} \),

\[
d_{\mathcal{Y}}(f(x_1), f(x_2)) \leq Cd_{\mathcal{X}}^\alpha(x_1, x_2).
\]

Prove that if \( \alpha > 0 \), \( f \) is continuous; if \( \alpha > 1 \), \( f \) is constant. (Hölder continuity with \( \alpha = 1 \) is also referred to as Lipschitz continuity.)

20. Construct a function \( f : \mathbb{R} \to \mathbb{R} \) which is continuous on all irrationals and discontinuous on all rationals. Prove that the opposite is impossible.

DIFFERENTIABILITY

21. Is there a function, differentiable on all irrationals and discontinuous on all rationals?

22. Suppose some sublevel set, \( \mathcal{F} = \{ x : f(x) \leq F \} \), of a differentiable function \( f : \mathcal{X} \to \mathbb{R} \) is compact, then \( f \) achieves its minimum at some \( x \in \mathcal{F} \), and its derivative at \( x \) vanishes.

23. Prove Taylor’s theorem using the mean value theorem.

24. If partial derivatives of \( f : \mathbb{R}^n \to \mathbb{R} \) are bounded in a neighborhood of \( x \), then \( f \) is continuous at \( x \).

25. Find a function discontinuous at the origin whose partial derivatives at the origin are nevertheless well-defined.

26. If there exists a function \( Df(x_0) : \mathcal{X} \to \mathcal{Y} \), such that for all \( x \in \mathcal{X} \),

\[
\lim_{\epsilon \to 0} \frac{\|f(x_0 + \epsilon x) - f(x_0) - \epsilon Df(x_0;x)\|}{\epsilon} = 0,
\]

it is called the directional (Gâteaux) derivative of \( f \) at \( x_0 \). Give examples of non-differentiable functions which are Gâteaux-differentiable. (Hint: this may happen if, e.g., \( Df(x_0;x) \) is not a linear map of \( x \).) Suppose \( Df(x_0) \) exists and is linear, would this imply Fréchet differentiability as well?

27. Give example of a function whose derivative at 0 is equal to 1, though the function itself is not invertible in any neighborhood of 0.
28. A function is of bounded variation iff it may be represented as a difference of two monotone-increasing functions.

29. **Integral test for convergence of series.** Suppose \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) is monotone-decreasing, then

\[
\sum_{n=1}^{\infty} f(n) \text{ converges iff } \int_1^\infty f(x) \, dx \text{ converges.}
\]

30. Prove that if \( \int_0^1 f(x) x^n \, dx = 0 \) for all \( n = 0, 1, 2 \ldots \) and \( f \) is continuous, then \( f \equiv 0 \) on \([0, 1]\).

31. Show by direct computation that

\[
\int_1^\infty \left( \int_1^\infty \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \right) \, dx = -\int_1^\infty \left( \int_1^\infty \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dx \right) \, dy = \frac{\pi}{4}.
\]

32. Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^2 \) with smooth boundary \( \partial \Omega \). Prove that

\[
\text{Vol}(\Omega) = \int \int_{\Omega} dx \, dy = \oint_{\partial \Omega} x \, dy = -\oint_{\partial \Omega} y \, dx = \frac{1}{2} \oint_{\partial \Omega} [x \, dy - y \, dx].
\]

### Sequences of Functions

33. Partial sums of power series and their derivatives (of all orders) converge uniformly on compact subsets of their open intervals of convergence.

34. For real-valued functions on a metric space \( \mathcal{X} \), define the *supremum norm*:

\[
\|f\| = \sup_{x \in \mathcal{X}} |f(x)|.
\]

The set of all continuous functions for which \( \|f\| < \infty \) is called \( C(\mathcal{X}) \). When is \( C(\mathcal{X}) \) a complete metric space with respect to the metric \( d(f, g) = \|f - g\| \)?

35. Suppose \( \{f_n(x)\} \) is a sequence of differentiable functions converging uniformly to \( f(x) \). Give an example illustrating that \( f(x) \) need not be differentiable. Give an example illustrating that the derivatives \( f_n'(x) \) need not converge. Suppose that \( f(x) \) is differentiable and \( f_n'(x) \) converge point-wise, show that the equality \( \lim_{n \to \infty} f_n'(x) = f'(x) \) need not hold.

36. **Peano’s existence theorem.** Suppose \( f : \mathbb{R}^2 \to \mathbb{R} \) is continuous in a neighborhood of \((x_0, y_0)\). Then there exists a function \( y(x) \), such that \( y(x_0) = y_0 \) and \( y'(x) = f(x, y(x)) \). (Hint: construct Euler approximations to the solution of this differential equation and show that they constitute an equicontinuous family of functions.)